PENETRATION MECHANICS: PREDICTING THE LOCATION OF A VISCOPLASTIC BOUNDARY AND ITS EFFECT ON THE STRESSES

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Abstract—Under the action of an external force, a solid body B penetrates into another body. Body B is assumed to be of an incompressible, viscoplastic, Bingham material, consisting of a solid core surrounded by a zone of viscoplastic flow. As a first model, the problem is treated one-dimensionally in the space variable x as well as the time variable t.

By utilizing Green's functions, the location of the moving boundary $s(t)$, i.e. the boundary between the region of viscoplastic flow and the core, is expressed in terms of an integral equation, which may then be solved numerically.

The resulting numerical method works well in practice, as illustrated by two examples.

1. FORMULATION OF THE PROBLEM

A solid body B of width $2H$ penetrates into another body under the action of an external force. We assume that body B is an incompressible viscoplastic Bingham body, that is, it satisfies Bingham's law,

$$\tau^*(x^*, t^*) - \tau_0 = \pm \mu \frac{\partial \mu^*}{\partial x^*}(x^*, t^*), \tag{1}$$

where τ^* is the stress, τ_0 the yield stress, μ the coefficient of viscosity and u^* the velocity in the y-direction. The movement is in the y-direction only and is assumed to be independent of z and symmetric about the plane $x^* = H$ (see Fig. 1). (The starred variables represent the original units; we will replace them below by their nondimensional counterparts.)

Perhaps it is appropriate at this point to note that the physical condition depicted here is that of a penetrated body consisting of bituminous material, e.g. asphalt. While it is well recognized that such an application is not the most desirable goal of the mechanics community, nevertheless its solution will provide us with sufficient information and knowledge which one may subsequently be able to use in order to extend the analysis of this notoriously difficult problem to also include other materials which are physically more desirable. Moreover, such a solution will serve as a limit check for the complicated finite element codes presently available. Thus, the case of an incompressible viscoplastic Bingham material is a logical fountainhead for detailed theoretical study.

Returning next to our present analysis, the body B is divided into two parts

$$B_1 = \{x^* : 0 < x^* < s^*(t^*) \quad \text{or} \quad 2H - s^*(t^*) < x^* < 2H\},$$

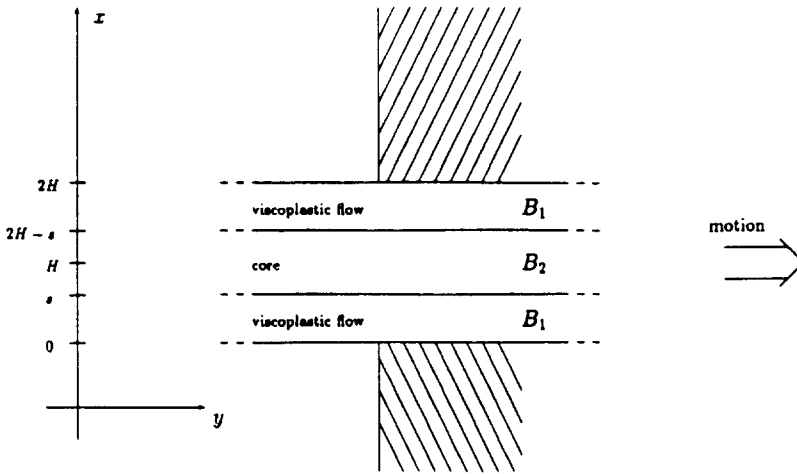$$B_2 = \{x^* : s^*(t^*) \leqslant x^* \leqslant 2H - s^*(t^*)\}. \tag{2}$$

Fig. 1. Geometrical configuration.

In B_1 (respectively B_2) the tangential stress is larger (respectively smaller) than the yield stress τ_0. We call B_1 the *zone of viscoplastic flow* and B_2 the *core*.

In the zone of viscoplastic flow, the velocity $u^*(x^*, t^*)$ satisfies the equation

$$\rho u_{t^*}^*(x^*, t^*) = \mu u_{x^*,x^*}^*(x^*, t^*) + g^*(t^*),\tag{3}$$

where g^* is the force per unit volume due to an external force acting in the y-direction and ρ is the density. Since the core is rigid, the velocity in it is

$$u_0^*(t^*) \overset{\text{def}}{=} u^*(s^*(t^*), t^*),\tag{4}$$

where it is assumed that $u_0^*(0) \neq 0$.

Before going any further, let us nondimensionalize the problem. We choose the half-width H of the body B as the characteristic length scale, τ_0 as the characteristic stress and τ_0/H as the characteristic force per unit volume. The characteristic time interval T will be specified later, separately for each application.

Thus, upon introducing the dimensionless variables

$$u = u^* T/H,$$
$$t = t^*/T,$$
$$x = x^*/H,$$
$$g = g^* H/\tau_0,$$
$$\tau = \tau^*/\tau_0,\tag{5}$$

Bingham's Law (1) becomes

$$\tau(x, t) - 1 = \pm \frac{1}{S} u_x(x, t)\tag{6}$$

and eqn (3) turns into

$$u_t(x, t) = \frac{1}{R} u_{xx}(x, t) + \frac{S}{R} g(t),\tag{7}$$

where

$$R = \frac{\rho H^2}{\mu T},$$

$$S = \frac{\tau_0 T}{\mu}. \tag{8}$$

R is the Reynolds number, i.e. the ratio of inertial to viscous forces. Similarly, S represents the ratio of external to viscous forces.

Due to symmetry, it is sufficient to consider eqn (7) in the domain $0 < x < s(t)$. On the moving boundary $s(t)$, the tangential stress is equal to the yield stress, so by Bingham's Law

$$u_x(s(t), t) = 0. \tag{9}$$

Considering the forces on the core [see e.g. Rubinstein (1970)], we find

$$\ddot{u}_0(t) = \frac{S}{R}g(t) - \frac{S}{R(1 - s(t))}. \tag{10}$$

Since by (9)

$$\ddot{u}_0(t) = \frac{d}{dt}[u(s(t), t)] = u_x(s(t), t)\dot{s}(t) + u_t(s(t), t) = u_t(s(t), t), \tag{11}$$

eqn (10) can be written as

$$u_t(s(t), t) = \frac{S}{R}g(t) - \frac{S}{R(1 - s(t))}. \tag{12}$$

Assuming continuity of the solution and all its derivatives up to the boundary and letting $x \nearrow s(t)$ in (7), we obtain

$$u_t(s(t), t) = \frac{1}{R}u_{xx}(s(t), t) + \frac{S}{R}g(t). \tag{13}$$

Upon comparison with (12), we must have

$$u_{xx}(s(t), t) = -\frac{S}{1 - s(t)}. \tag{14}$$

We assume we are given the boundary and initial values

$$u(0, t) = f(t),$$

$$u(x, 0) = \phi(x),$$

$$s(0) = b, \quad 0 < b < 1. \tag{15}$$

To make (9), (14) and (15) consistent, we must require that

$$\phi(0) = f(0),$$

$$\phi'(b) = 0,$$

$$\phi''(b) = -\frac{S}{1 - b}. \tag{16}$$

Notice that in this analysis we have for simplicity assumed that $s(0) > 0$. The case $s(0) = 0$ requires some special mathematical rigor which for the sake of brevity we will omit.

Perhaps it is appropriate at this point to comment on the difference between the present problem and the classical Stefan problem. For the classical Stefan problem, the location of the moving boundary $x = s(t)$ is governed by the velocity u as well as its derivative with respect to x, whereas in the present problem it is also governed by the time derivative, i.e. an additional constraint which makes the solution even more difficult.

Before engaging in the details of the construction of the solution, let us summarize the problem we are trying to solve. Given a time $T_{max} > 0$, we are looking for a pair of functions $u(x, t)$, $s(t)$ so that

- $s(t)$ is Lipschitz continuous on $(0, T_{max}]$;
- u and u_x are continuous for $0 \leqslant x \leqslant s(t)$, $0 \leqslant t \leqslant T_{max}$;
- u_{xx}, u_t are continuous in $0 \leqslant x \leqslant s(t)$ for $0 < t < T_{max}$;
- u satisfies the equation

$$u_t(x, t) = \frac{1}{R} u_{xx}(x, t) + \frac{S}{R} g(t) \tag{17}$$

in $0 < x < s(t)$, $0 < t \leqslant T_{max}$;
- on the moving boundary $s(t)$, u satisfies

$$u_t(s(t), t) = \frac{S}{R} g(t) - \frac{S}{R(1 - s(t))},$$

$$u_x(s(t), t) = 0,$$

$$u_{xx}(s(t), t) = - \frac{S}{1 - s(t)}, \tag{18}$$

for $0 < t \leqslant T_{max}$;
- u and s satisfy the boundary and initial conditions

$$s(0) = b, \quad 0 < b < 1,$$

$$u(x, 0) = \phi(x),$$

$$u(0, t) = f(t), \tag{19}$$

with compatibility conditions

$$\phi(0) = f(0),$$
$$\phi'(b) = 0,$$
$$\phi''(b) = - \frac{S}{1 - b}. \tag{20}$$

This problem has been discussed before by the authors in Ang *et al.* (1989). A related problem is solved by similar methods in Ang *et al.* (1988).

2. REFORMULATION OF THE PROBLEM

We shall reformulate the problem as an integral equation in $r(t) \stackrel{def}{=} s(t)$, which can be solved by successive approximation, using the contraction principle. For this purpose we shall require some regularity conditions on the initial and boundary data:

- $f(t)$ is continuous, $g(t)$ is C^1 on $t \geqslant 0$;
- $\phi(x)$ is C^2 on $(0, b)$, and the left-hand derivative $\phi''(b)$ exists.

Put $v = u_t$. The equations for $v(x, t)$ can be derived from the corresponding equations for u by differentiation. Thus, from (7),

$$v_t(x, t) = \frac{1}{R} v_{xx}(x, t) + \frac{S}{R} \dot{g}(t).$$ (21)

Differentiating (9) with respect to t gives

$$0 = \frac{d}{dt} [u_x(s(t), t)] = u_{xx}(s(t), t)\dot{s}(t) + v_x(s(t), t),$$ (22)

thus by (14)

$$v_x(s(t), t) = S \frac{\dot{s}(t)}{1 - s(t)}.$$ (23)

Equation (12) becomes simply

$$v(s(t), t) = \frac{S}{R} g(t) - \frac{S}{R(1 - s(t))}.$$ (24)

Equations (7) at $t = 0$ and (15) give

$$v(x, 0) = u_t(x, 0) = \frac{1}{R} \phi''(x) + \frac{S}{R} g(0) \stackrel{\text{def}}{=} \psi(x),$$ (25)

while the boundary condition at $x = 0$ is

$$v(0, t) = \dot{f}(t).$$ (26)

The compatibility conditions are

$$\dot{f}(0) = \psi(0)$$

$$\psi(b) = \frac{S}{R} g(0) - \frac{S}{R(1 - b)}.$$ (27)

Summing up the above equations, v must satisfy

$$v_t(x, t) = \frac{1}{R} v_{xx}(x, t) + \frac{S}{R} \dot{g}(t),$$

$$v(s(t), t) = \frac{S}{R} g(t) - \frac{S}{R(1 - s(t))},$$

$$v_x(s(t), t) = S \frac{\dot{s}(t)}{1 - s(t)},$$

$$v(x, 0) = \psi(x),$$

$$v(0, t) = \dot{f}(t),$$

$$\psi(b) = \frac{S}{R} g(0) - \frac{S}{R(1 - b)},$$

$$\dot{f}(0) = \psi(0).$$ (28)

Assume for now that $s(t)$ is C^1 on $[0, \sigma]$.†

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† We will justify this assumption later.

Let $k = R^{1/2}$. We define the Green's functions

$$K(x, t; \xi, \tau) = \frac{k}{2\sqrt{\pi}} \frac{1}{\sqrt{t-\tau}} \exp\left(- \frac{k^2(x-\xi)^2}{4(t-\tau)} \right),$$

$$G(x, t; \xi, \tau) = K(x, t; \xi, \tau) - K(x, t; -\xi, \tau),$$

$$N(x, t; \xi, \tau) = K(x, t; \xi, \tau) + K(x, t; -\xi, \tau), \tag{29}$$

for

$$0 < x < s(t), \quad 0 < \xi < s(\tau), \quad 0 < \tau < t. \tag{30}$$

We will use the following properties, which are easy to verify

$$G_{x\xi} = k^2 N_\tau$$

$$G_x = - N_\xi$$

$$N(x, t; \xi, t) = 0. \tag{31}$$

Thus, let $v(\xi, \tau, s(\tau))$ be a solution of (28) with (x, t) replaced by (ξ, τ). Integrating the identity

$$(Gv_\xi - G\xi_v)_\xi - k^2(Gv)_\tau = -SG\dot{g} \tag{32}$$

over the region $\{(\xi, \tau) : 0 \leqslant \xi \leqslant s(\tau), \varepsilon \leqslant \tau \leqslant t - \varepsilon\}$, applying Green's identity and letting $\varepsilon \to 0$, we obtain

$$v(x, t) = \int_0^b \psi(\xi)G(x, t; \xi, 0)\, d\xi - \frac{1}{R} \int_0^t v_0(\tau)G_\xi(x, t; s(\tau), \tau)\, d\tau$$

$$+ \frac{1}{R} \int_0^t v_x(s(\tau), \tau)G(x, t; s(\tau), \tau)\, d\tau + \int_0^t v_0(\tau)G(x, t; s(\tau), \tau)\dot{s}(\tau)\, d\tau$$

$$+ \frac{1}{R} \int_0^t f(\tau)G_\xi(x, t; 0, \tau)\, d\tau + \frac{S}{R} \int_0^t \int_0^{s(\tau)} G(x, t; \xi, \tau)\, d\xi \dot{g}(\tau)\, d\tau, \tag{33}$$

where we have put

$$v_0(t) = v(s(t), t) \tag{34}$$

and have used the identity

$$d\xi = \dot{s}(\tau)\, d\tau \quad \text{on} \quad s(t). \tag{35}$$

Take the x-derivative of both sides of (33) to get

$$v_x(x, t) = \int_0^b \psi(\xi)G_x(x, t; \xi, 0)\, d\xi - \frac{1}{R} \int_0^t v_0(\tau)G_{x\xi}(x, t; s(\tau), \tau)\, d\tau$$

$$+ \frac{1}{R} \int_0^t v_x(s(\tau), \tau)G_x(x, t; s(\tau), \tau)\, d\tau + \int_0^t v_0(\tau)G_x(x, t; s(\tau), \tau)\dot{s}(\tau)\, d\tau$$

$$+ \frac{1}{R} \int_0^t f(\tau)G_{x\xi}(x, t; 0, \tau)\, d\tau + \frac{S}{R} \int_0^t \int_0^{s(\tau)} G_x(x, t; \xi, \tau)\, d\xi \dot{g}(\tau)\, d\tau. \tag{36}$$

Integrate the various terms on the right-hand side by parts

$$\int_0^b \psi(\xi)G_r(x,t;\xi,0)\,d\xi = -\int_0^b \psi(\xi)N_\xi(x,t;\xi,0)\,d\xi$$

$$= \psi(0)N(x,t;0,0) - \psi(b)N(x,t;b,0)$$

$$+ \int_0^b \psi'(\xi)N(x,t;\xi,0)\,d\xi. \tag{37}$$

$$-\frac{1}{R}\int_0^t v_0(\tau)G_{r\xi}(x,t;s(\tau),\tau)\,d\tau$$

$$= -\int_0^t v_0(\tau)N_\tau(x,t;s(\tau),\tau)\,d\tau$$

$$= -\int_0^t v_0(\tau)\left\{\frac{d}{d\tau}[N(x,t;s(\tau),\tau)] - N_\xi(x,t;s(\tau),\tau)r(\tau)\right\}d\tau$$

$$= v_0(0)N(x,t;b,0) + \int_0^t \dot{v}_0(\tau)N(x,t;s(\tau),\tau)\,d\tau$$

$$+ \int_0^t v_0(\tau)N_x(x,t;s(\tau),\tau)r(\tau)\,d\tau \tag{38}$$

$$\frac{1}{R}\int_0^t f(\tau)G_{x\xi}(x,t;0,\tau)\,d\tau = \int_0^t f(\tau)N_\tau(x,t;0,\tau)\,d\tau$$

$$= -f(0)N(x,t;0,0) - \int_0^t \dot{f}(\tau)N(x,t;0,\tau)\,d\tau, \tag{39}$$

$$\int_0^{s(t)} G_x(x,t;\xi,\tau)\,d\xi = -\int_0^{s(t)} N_\xi(x,t;\xi,\tau)\,d\xi$$

$$= -N(x,t;s(\tau),\tau) + N(x,t;0,\tau), \tag{40}$$

so

$$\frac{S}{R}\int_0^t \int_0^{s(t)} G_x(x,t;\xi,\tau)\,d\xi\,\dot{g}(\tau)\,d\tau = -\frac{S}{R}\int_0^t \dot{g}(\tau)\{N(x,t;s(\tau),\tau) - N(x,t;0,\tau)\}\,d\tau. \tag{41}$$

We end up with

$$v_r(x,t) = \int_0^b \psi'(\xi)N(x,t;\xi,0)\,d\xi + \int_0^t \dot{v}_0(\tau)N(x,t;s(\tau),\tau)\,d\tau$$

$$+ \frac{1}{R}\int_0^t v_x(s(\tau),\tau)G_x(x,t;s(\tau),\tau)\,d\tau - \int_0^t \dot{f}(\tau)N(x,t;0,\tau)\,d\tau$$

$$- \frac{S}{R}\int_0^t \dot{g}(\tau)\{N(x,t;s(\tau),\tau) - N(x,t;0,\tau)\}\,d\tau. \tag{42}$$

Now, let $x \nearrow s(t)$ and use a lemma from Friedman (1964).

Lemma 2.1 (Friedman). Let $\rho(t)$ ($0 \leqslant t \leqslant \sigma$) be a continuous function and let $s(t)$ ($0 \leqslant t \leqslant \sigma$) satisfy a Lipschitz condition. Then, for every $0 < t \leqslant \sigma$,

$$\lim_{x \to s(t)-0} \frac{\partial}{\partial x} \int_0^t \rho(\tau) K(x, t; s(\tau), \tau) \, d\tau = \frac{k^2}{2} \rho(t) + \int_0^t \rho(\tau) \left[\frac{\partial}{\partial x} K(x, t; s(\tau), \tau) \right]_{x = s(t)} \, d\tau, \qquad (43)$$

where K is as in (29).

This gives

$$v_x(s(t), t) = \int_0^b \psi'(\xi) N(s(t), t; \xi, 0) \, d\xi + \int_0^t \dot{v}_0(\tau) N(s(t), t; s(\tau), \tau) \, d\tau$$

$$+ \tfrac{1}{2} v_x(s(t), t) + \frac{1}{R} \int_0^t v_x(s(\tau), \tau) G_x(s(t), t; s(\tau), \tau) \, d\tau$$

$$- \int_0^t f'(\tau) N(s(t), t; 0, \tau) \, d\tau$$

$$- \frac{S}{R} \int_0^t \dot{g}(\tau) \{ N(s(t), t; s(\tau), \tau) - N(s(t), t; 0, \tau) \} \, d\tau. \qquad (44)$$

or

$$\tfrac{1}{2} v_x(s(t), t) = \int_0^b \psi'(\xi) N(s(t), t; \xi, 0) \, d\xi + \int_0^t \dot{v}_0(\tau) N(s(t), t; s(\tau), \tau) \, d\tau$$

$$+ \frac{1}{R} \int_0^t v_x(s(\tau), \tau) G_x(s(t), t; s(\tau), \tau) \, d\tau$$

$$- \int_0^t f'(\tau) N(s(t), t; 0, \tau) \, d\tau$$

$$- \frac{S}{R} \int_0^t \dot{g}(\tau) \{ N(s(t), t; s(\tau), \tau) - N(s(t), t; 0, \tau) \} \, d\tau. \qquad (45)$$

Define

$$r(t) = \dot{s}(t), \qquad (46)$$

so that

$$s(t) = b + \int_0^t r(\tau) \, d\tau, \qquad (47)$$

and recall (23) and (24)

$$v_x(s(t), t) = S \frac{r(t)}{1 - s(t)},$$

$$v_0(t) = \frac{S}{R} g(t) - \frac{S}{R(1 - s(t))},$$

$$\dot{v}_0(t) = \frac{S}{R} \dot{g}(t) - \frac{S r(t)}{R(1 - s(t))^2}. \qquad (48)$$

The left-hand side of (45) equals

$$\tfrac{1}{2} v_x(s(t), t) = \frac{S}{2} \frac{r(t)}{1 - s(t)}. \qquad (49)$$

while the right-hand side becomes

$$\int_0^h \psi'(\xi)N(s(t),t;\xi,0)\,d\xi$$

$$+ \frac{S}{R}\int_0^t \left[\dot{g}(\tau) - \frac{r(\tau)}{(1-s(\tau))^2}\right]N(s(t),t;s(\tau),\tau)\,d\tau$$

$$+ \frac{S}{R}\int_0^t \left[\frac{r(\tau)}{1-s(\tau)}\right]G_x(s(t),t;s(\tau),\tau)\,d\tau$$

$$- \int_0^t \dot{f}(\tau)N(s(t),t;0,\tau)\,d\tau$$

$$- \frac{S}{R}\int_0^t \dot{g}(\tau)\{N(s(t),t;s(\tau),\tau) - N(s(t),t;0,\tau)\}\,d\tau. \tag{50}$$

After some minor simplifications, we end up with

$$r(t) = \frac{2}{S}(1-s(t))B(r(t)), \tag{51}$$

where

$$B(r(t)) = \int_0^h \psi'(\xi)N(s(t),t;\xi,0)\,d\xi - \frac{S}{R}\int_0^t \frac{r(\tau)}{(1-s(\tau))^2}N(s(t),t;s(\tau),\tau)\,d\tau$$

$$+ \frac{S}{R}\int_0^t \frac{r(\tau)}{1-s(\tau)}G_x(s(t),t;s(\tau),\tau)\,d\tau$$

$$- \int_0^t \left[\dot{f}(\tau) - \frac{S}{R}\dot{g}(\tau)\right]N(s(t),t;0,\tau)\,d\tau. \tag{52}$$

It can be shown that there exist $M > 0$ and $\sigma > 0$ such that the right-hand side of (51) defines a contraction on $B_\sigma(0,M)$, the closed ball of radius M, center 0 in the space of continuous functions on $[0,\sigma]$.

Thus, for small values of t, iteration of (51) will produce a solution $r(t)$. (In the numerical experiments, no limit on the values of t was found; the method converged in all cases.) In addition to providing the basis for a numerical method, this justifies the smoothness assumptions on $s(t)$ made earlier.

We will use eqn (51) as the basis for a numerical method, similar to the one used in Ang *et al.* (1988).

3. NUMERICAL RESULTS

Formula (51) forms the basis of a numerical method as follows.

Let t_i, $i = 0,1,2,\ldots$, be equally spaced points in the t direction. Given a guess for $r(t)$, we can calculate a guess for $s(t)$ from (47), then $B(r(t))$ from (52), and finally an updated $r(t)$ from (51).

Once $s(t)$ is known, we can calculate $u_x(x,t)$ for any value of (x,t) from

$$u_x(x,t) = \int_0^h \phi'(\xi)N(x,t;\xi,0)\,d\xi - \frac{S}{R}\int_0^t \frac{1}{1-s(\tau)}N(x,t;s(\tau),\tau)\,d\tau$$

$$+ \int_0^t \left[\frac{S}{R}g(\tau) - f(\tau)\right]N(x,t;0,\tau)\,d\tau. \tag{53}$$

This equation is similar to eqn (42) and derived the same way. Finally, we can calculate $u(x, t)$ from

$$u(x, t) = f(t) + \int_0^x u_x(\xi, t) \, d\xi. \tag{54}$$

As a check, $u(s(t), t)$ can also be calculated from

$$u(s(t), t) = \int_0^t u(s(\tau), \tau) \, d\tau = \frac{S}{R} \int_0^t \left[g(\tau) - \frac{1}{1 - s(\tau)} \right] d\tau. \tag{55}$$

A value for $r(0)$ is provided by the third formula in (28)

$$r(0) = \frac{1}{S}(1 - b)\psi'(b). \tag{56}$$

The first iteration starts out with this value for $r(t_0)$ and $r(t_1)$.

At the ith iteration, we only used values of r at $t_0, \ldots, t_i$, since good guesses of $r(t)$ for large t were not available. For the $(i + 1)$ iteration, we added a new point $r(t_{i+1})$, with initial guess

$$r(t_{i+1}) = 2r(t_i) - r(t_{i-1}). \tag{57}$$

Only $r(t_i)$ had to be calculated at the ith step, since the previous values of r were already known and not affected by later values.

Except near the start, four to five iterations per point, combined with extrapolation, were sufficient for convergence.

We used free spline interpolation to calculate r, s at intermediate points, and routines from QUADPACK for numerical integration. Note that the second and third integrals in (52) are singular, but the type of singularity $[(t - \tau)^{-1/2}]$ is known exactly and can be easily handled.

As an aid in selecting appropriate test problems, we note the existence of special steady flow solutions of the form

$$u(x, t) = f_0 + \frac{S}{2} g_0 [b^2 - (x - b)^2], \tag{58}$$

where $f(t) \equiv f_0$, $g(t) \equiv g_0$, $s(t) \equiv b$ and $b = 1 - 1/g_0$.

In all numerical experiments, we used $S = R = 1$, $f(t) \equiv 0$.

Example 1. We started with the steady flow solution corresponding to $g(t) \equiv 2$, that is, $b = 0.5$, $\phi(x) = 0.25 - (x - 0.25)^2$. We then set $g(t) \equiv 0$, corresponding to an abrupt vanishing of the external force. Thus, the motion is dominated by viscous forces.

Figure 2 shows plots of the moving boundary $s(t)$, the core velocity $u_0(t)$ and of the velocity $u(x, t)$ and the stress $\tau(x, t)$ in the zone of viscoplastic flow for various fixed values of x and t. We used a small time step of 0.001 to produce smooth curves; the results for larger time steps are in excellent agreement. As one would expect, the core expands rapidly until it reaches the boundary $x = 0$.

If $g(t)$ is taken to be 2 initially, then dropped to 0, we obtain time-shifted versions of the same curves. This indicates that the method can handle discontinuous external forces easily. The last integral in (52), must, of course, be modified to account for the delta function behavior of $\dot{g}(t)$.

Example 2. We used an external force

$$g(t) = 2[1 - \tfrac{49}{55} e^{-t}]. \tag{59}$$
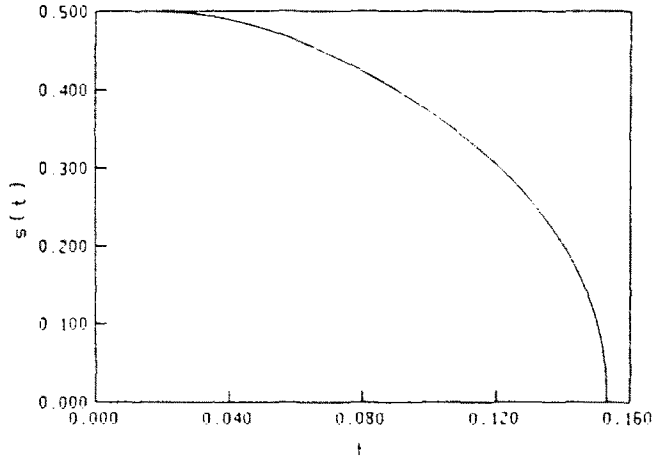
Moving Boundary

Fig. 2a. Example 1 : Moving boundary.
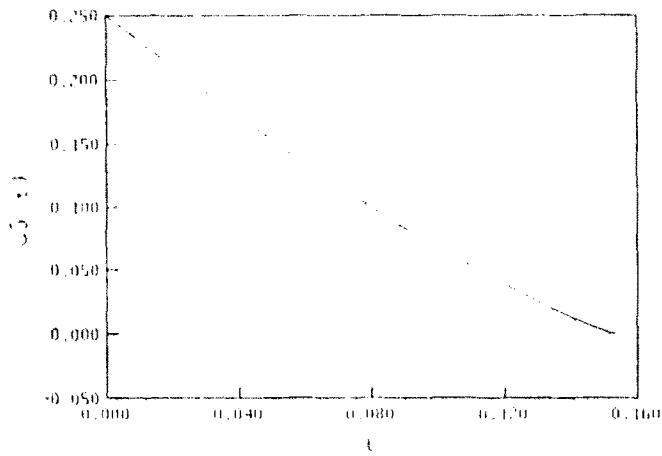
Core Velocity

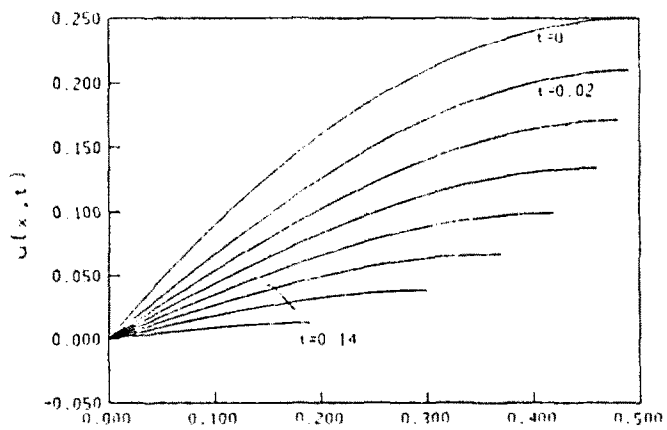Fig. 2b. Example 1 : Core velocity.

Velocity Profiles for fixed t

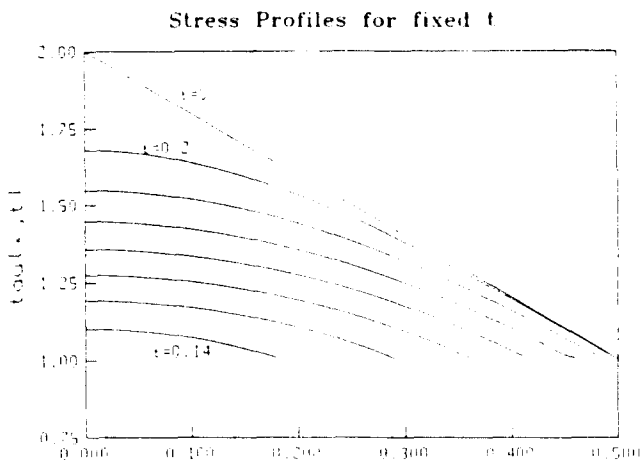Fig. 2c. Example 1 : Velocity in flow zone for $t = 0$ to $t = 0.14$ in steps of 0.02.

Stress Profiles for fixed t

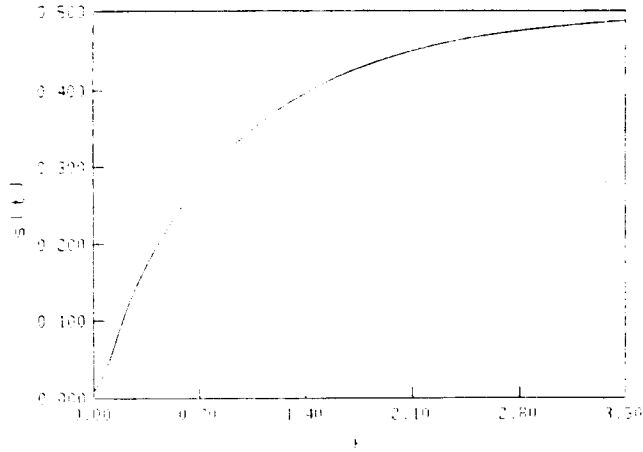Fig. 2d. Example 1 : Stress in flow zone for $t = 0$ to $t = 0.14$ in steps of 0.02.

Velocity Profiles for fixed x

Fig. 2e. Example 1 : Velocity in flow zone for $x = 0$ to $x = 0.4$ in steps of 0.1.
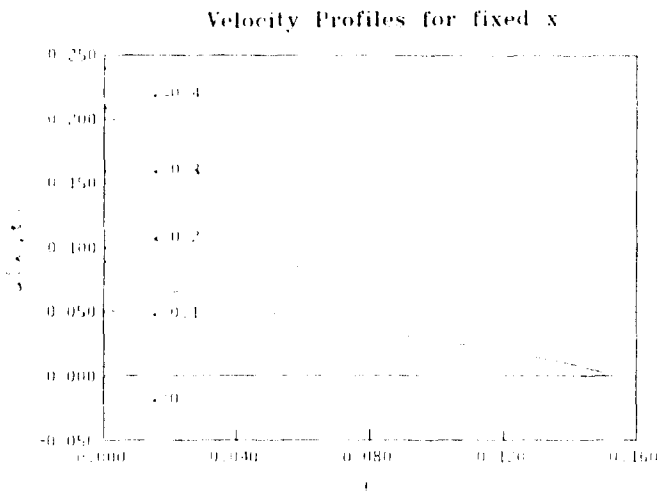
Stress Profiles for fixed x
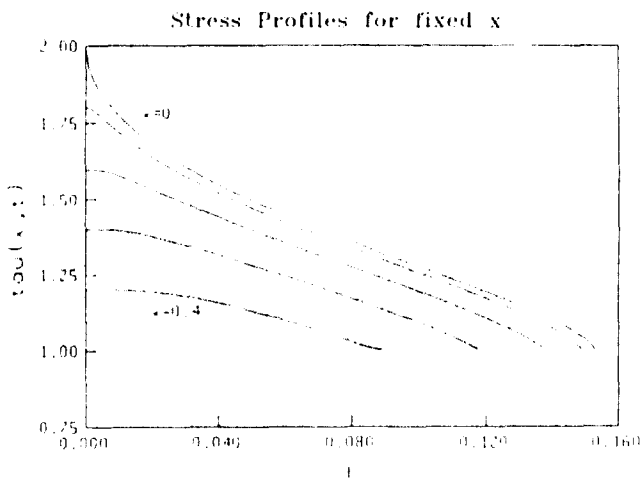
Fig. 2f. Example 1 : Stress in flow zone for $x = 0$ to $x = 0.4$ in steps of 0.1.

Moving Boundary

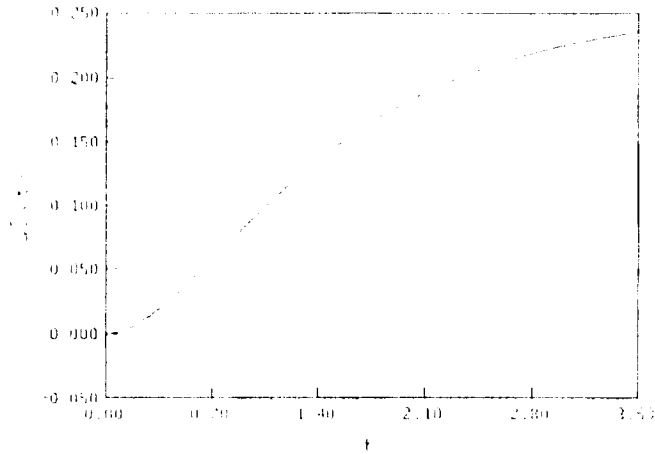Fig. 3a. Example 2: Moving boundary.

Core Velocity

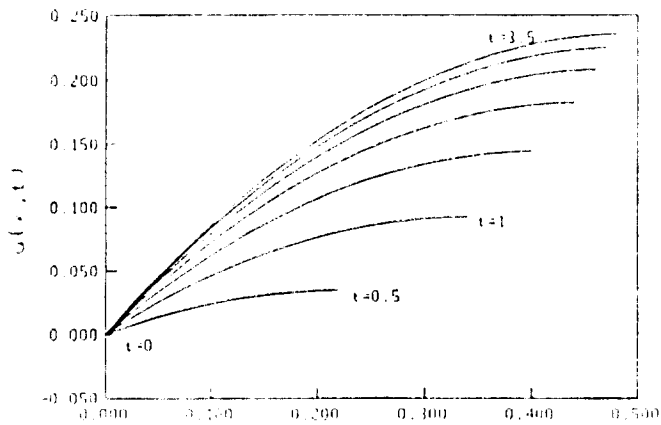Fig. 3b. Example 2: Core velocity.

Velocity Profiles for fixed t

Fig. 3c. Example 2: Velocity in flow zone for $t = 0$ to $t = 3.5$ in steps of 0.5.
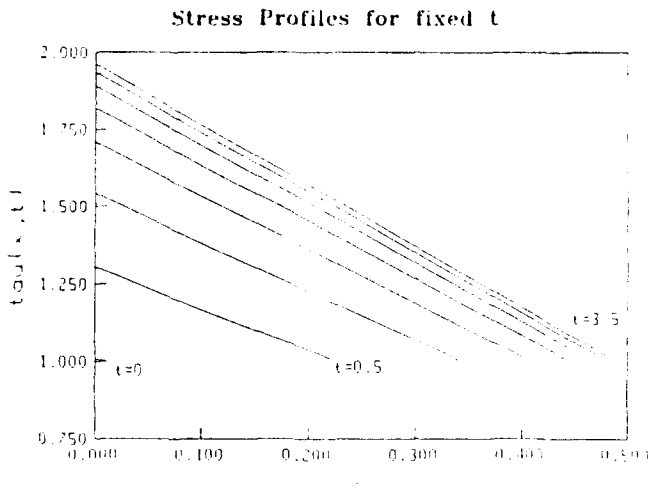
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Stress Profiles for fixed t

tau(x,t)

t=0   t=0.5   t=3.5

Fig. 3d. Example 2: Stress in flow zone for $t = 0$ to $t = 3.5$ in steps of 0.5.

Velocity Profiles for fixed x

x=0.4
x=0.3
x=0.2
x=0.1
x=0

Fig. 3e. Example 2: Velocity in flow zone for $x = 0$ to $x = 0.4$ in steps of 0.1.

Stress Profiles for fixed x

tau(x,t)

x=0
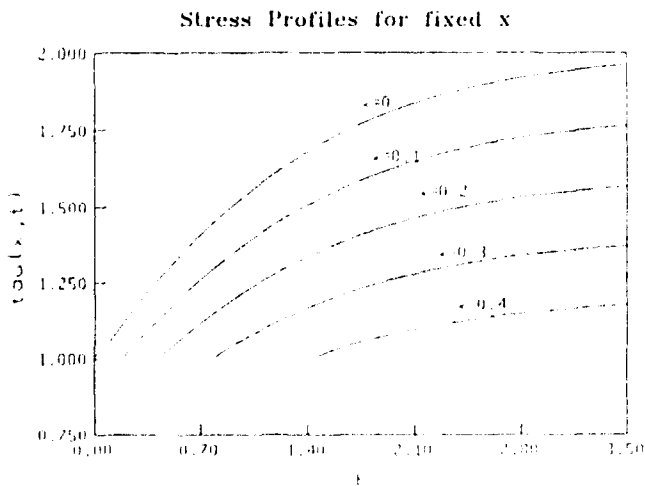x=0.1
x=0.2
x=0.3
x=0.4

t
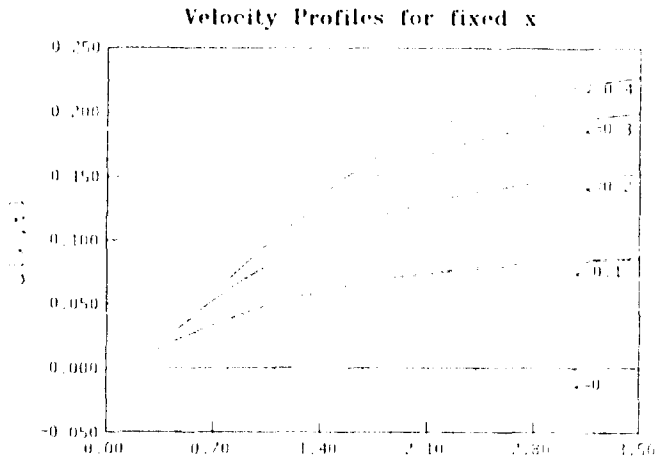
Fig. 3f. Example 2: Stress in flow zone for $x = 0$ to $x = 0.4$ in steps of 0.1.

and initial conditions corresponding to the steady state for $g_0 = 1.0101...$. The initial location of the moving boundary is $h = 0.01$, very close to 0. (The case $h = 0$ needs further investigation, as previously noted.)

The external force $g(t)$ increases rapidly to a limiting value of 2, so we would expect the moving boundary to approach the value 0.5. The curves in Fig. 3 were generated with a time step of 0.025.

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